

# Space-time fractional Zener wave equation

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## Abstract

Space-time fractional Zener wave equation, describing viscoelastic materials obeying the time-fractional Zener model and the space-fractional strain measure, is derived and analyzed. This model includes waves with finite speed, as well as non-propagating disturbances. The existence and the uniqueness of the solution to the generalized Cauchy problem are proved. Special cases are investigated and numerical examples are presented.

Key words: fractional Zener model, fractional strain measure, Laplace and Fourier transforms, Cauchy problem, generalized solution

## 1 Introduction

The aim of this study is a class of generalized wave equation. Wave equation can be generalized within the theory of fractional calculus by replacing the second order derivative (space and/or time) with the fractional ones, as done in [4, 5, 12, 13, 16, 17, 18]. Space-time fractional Zener wave equation represents a generalization of the classical wave equation obtained as a system consisting of the equation of motion of the deformable (one-dimensional) body, the time-fractional Zener constitutive equation and the space-fractional strain measure. Our generalization is done by the fractionalization in both space and time variable on the ground of the physically acceptable concepts. More details on the formulation and the mechanical background will be given in this section, which finishes with the remark related to the analysis of our generalization of the wave equation.

In Section 2 we show the existence and uniqueness of solution to the space-time fractional Zener wave equation (9), (11), (12). For this purpose we use the Fourier and Laplace transforms in the spaces of distributions, simplifying the procedure, in a way that we have to prove the absolute convergence of certain double integrals. The analysis presented in Section 2.2 concerning the properties of solution to the space-time fractional Zener wave equation implies that the solution kernel is

$$P(x, t) = I(x, t) - \left( \frac{\partial}{\partial t} J_1(x, t) + \frac{\partial^2}{\partial t^2} J_2(x, t) \right) e^{s_0 t}, \quad x \in \mathbb{R}, \quad t > 0,$$

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where  $I$ ,  $J_1$  and  $J_2$  are continuous bounded function with respect to the space variable  $x$  and continuous exponentially bounded functions with respect to the time variable  $t$ , see Theorem 2. On the other hand, in Section 2.3 we show, by the regularization and quite different estimates in comparison to those used in Section 2.2, that the solution to (9), (11), (12) is given by a distributional limit of a net of approximated solutions, which are continuous with respect to  $x \in \mathbb{R}$ ,  $t > 0$ , bounded with respect to  $x \in \mathbb{R}$  and exponentially bounded with respect to  $t > 0$ . Results of Section 2 are justified in Section 3 by discussing the influence of parameters  $\alpha$  and  $\beta$  (orders of the time and space fractional derivatives) on the solution to (9), (11), (12) and in Section 4 by the numerical examples. Mathematical background is given in Appendix A.

## 1.1 Model

Recall, the classical wave equation describes the waves that occur in elastic medium. It is obtained from the equations of the deformable body, see [1]. The wave equation can be written in the form of a system which consists of three equations: equation of motion, constitutive equation and strain measure. Unknown functions depending on time,  $t > 0$ , and space,  $x \in \mathbb{R}$ , variables are: displacement  $u$ , stress  $\sigma$  and strain  $\varepsilon$ . We consider an infinite viscoelastic rod (one-dimensional body), positioned along  $x$ -axis, that is not under influence of body forces. Then, the equation of motion reads

$$\partial_x \sigma(x, t) = \rho \partial_t^2 u(x, t), \quad x \in \mathbb{R}, \quad t > 0, \quad (1)$$

where  $\rho > 0$  denotes the (constant) density of the rod. The constitutive equation gives the relation between stress and strain, and in the case of elastic media it is the Hooke law. Since we consider waves occurring in viscoelastic media, we chose the constitutive equation to be the time-fractional Zener model

$$\sigma(x, t) + \tau_\sigma {}^C D_t^\alpha \sigma(x, t) = E(\varepsilon(x, t) + \tau_\varepsilon {}^C D_t^\alpha \varepsilon(x, t)), \quad x \in \mathbb{R}, \quad t > 0, \quad (2)$$

where  $E$  is the generalized Young modulus (measured in  $\frac{\text{Pa}}{\text{m}^{1-\beta}}$ ),  $\tau_\sigma$  and  $\tau_\varepsilon$  are generalized relaxation times (measured in  $\text{s}^\alpha$ ) with (thermodynamical) restriction  $0 < \tau_\sigma < \tau_\varepsilon$ . All three parameters are assumed to be constant. The operator  ${}_0^C D_t^\alpha$  denotes the left Caputo operator of fractional differentiation of order  $\alpha \in [0, 1)$ , see Appendix A. For  $\alpha = 0$ , the constitutive equation (2) reduces to the Hooke law

$$\sigma = E_r \varepsilon, \quad \text{with} \quad E_r = E \frac{1 + \tau_\varepsilon}{1 + \tau_\sigma}.$$

For  $\alpha = 1$ , the constitutive equation (2) reduces to the classical Zener model. For more details on fractional derivatives see [24, 26]. We refer to [25] for a review on the fractional models in viscoelasticity and to [2] for a systematic analysis of the thermodynamical restrictions on parameters in such models. The strain measure gives connection between strain and displacement. In the classical set up, strain measure describes local deformations and it reads:  $\varepsilon = \partial_x u$ . Since we consider non-local effects in material, we use the fractional model of strain measure

$$\varepsilon(x, t) = \mathcal{E}_x^\beta u(x, t), \quad x \in \mathbb{R}, \quad t > 0, \quad (3)$$

where  $\mathcal{E}_x^\beta$  denotes the symmetrized Caputo fractional derivative of order  $\beta \in [0, 1)$ , see Appendix A. For  $\beta = 1$  we obtain the classical strain measure. Regarding the fractionalization of the strain measure, we follow the approach presented in [7], where the symmetrized fractional derivative is introduced in order to describe the non-local effects of the material. Note that in [3], the same type of the fractional derivative is used in the framework of the heat conduction problem of the space-time fractional Cattaneo type equation.

One may also treat the non-locality in viscoelastic media by the different approach. Namely, contrary to (3), one may retain the classical strain measure and introduce the non-locality in the constitutive equation. In the classical setting it was done by Eringen, [11]. In the framework of the fractional calculus this approach is followed in [8, 9, 21, 22, 23]. The wave equation, obtained from a system consisting of the equation of motion, fractional Eringen-type constitutive equation and classical strain measure, is studied in [10, 27].

The initial conditions corresponding to system (1) - (3) are

$$u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = v_0(x), \quad \sigma(x, 0) = 0, \quad \varepsilon(x, 0) = 0, \quad x \in \mathbb{R}, \quad (4)$$

where  $u_0$  and  $v_0$  are initial displacement and velocity, while the boundary conditions are

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} \sigma(x, t) = 0, \quad t > 0. \quad (5)$$

Note that boundary conditions (5) are the natural choice for the case of the unbounded domain, while in the case of the bounded domain there can be a large variety of different boundary conditions depending on the type of problem one faces with. In the case of the local, time-fractional wave equation on a bounded domain we refer to [2, 6, 25] and references therein.

## 1.2 System

Introducing the dimensionless quantities

$$\begin{aligned} \bar{x} &= x \left( (\tau_\varepsilon)^{\frac{2}{\alpha}} \frac{\rho}{E} \right)^{-\frac{1}{1+\beta}}, \quad \bar{t} = t (\tau_\varepsilon)^{-\frac{1}{\alpha}}, \quad \bar{u} = u \left( (\tau_\varepsilon)^{\frac{2}{\alpha}} \frac{\rho}{E} \right)^{-\frac{1}{1+\beta}}, \\ \bar{\sigma} &= \frac{\sigma}{E} \left( (\tau_\varepsilon)^{\frac{2}{\alpha}} \frac{\rho}{E} \right)^{-\frac{1-\beta}{1+\beta}}, \quad \bar{\varepsilon} = \varepsilon \left( (\tau_\varepsilon)^{\frac{2}{\alpha}} \frac{\rho}{E} \right)^{-\frac{1-\beta}{1+\beta}}, \quad \tau = \frac{\tau_\sigma}{\tau_\varepsilon}, \\ \bar{u}_0 &= u_0 \left( (\tau_\varepsilon)^{\frac{2}{\alpha}} \frac{\rho}{E} \right)^{-\frac{1}{1+\beta}}, \quad \bar{v}_0 = v_0 (\tau_\varepsilon)^{\frac{1}{\alpha}} \left( (\tau_\varepsilon)^{\frac{2}{\alpha}} \frac{\rho}{E} \right)^{-\frac{1}{1+\beta}} \end{aligned}$$

in (1) - (3) and omitting bar we obtain

$$\partial_x \sigma(x, t) = \partial_t^2 u(x, t), \quad x \in \mathbb{R}, \quad t > 0, \quad (6)$$

$$\sigma(x, t) + \tau {}^C D_t^\alpha \sigma(x, t) = \varepsilon(x, t) + {}^C D_t^\alpha \varepsilon(x, t), \quad x \in \mathbb{R}, \quad t > 0, \quad (7)$$

$$\varepsilon(x, t) = \mathcal{E}_x^\beta u(x, t), \quad x \in \mathbb{R}, \quad t > 0. \quad (8)$$

System (6) - (8) can be reduced to the space-time fractional Zener wave equation

$$\partial_t^2 u(x, t) = L_t^\alpha \partial_x \mathcal{E}_x^\beta u(x, t), \quad x \in \mathbb{R}, \quad t > 0, \quad (9)$$

where  $L_t^\alpha$  is a linear operator (of convolution type) given by

$$L_t^\alpha = \mathcal{L}^{-1} \left[ \frac{1 + s^\alpha}{1 + \tau s^\alpha} \right] *_t = \left( \frac{1}{\tau} \delta(t) + \left( \frac{1}{\tau} - 1 \right) e'_\alpha(t) \right) *_t, \quad t > 0, \quad (10)$$

and  $\mathcal{L}^{-1}$  denotes the inverse Laplace transform, see Appendix A. The dimensionless quantities give that initial and boundary conditions, (5) and (4), for the space-time fractional Zener wave equation (9) again become

$$u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = v_0(x), \quad \sigma(x, 0) = 0, \quad \varepsilon(x, 0) = 0, \quad x \in \mathbb{R}, \quad (11)$$

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} \sigma(x, t) = 0, \quad t > 0. \quad (12)$$

The procedure of obtaining (9) is as follows. Applying the Laplace transform to (7) with respect to time variable  $t$ , one obtains

$$(1 + \tau s^\alpha) \tilde{\sigma}(x, s) = (1 + s^\alpha) \tilde{\varepsilon}(x, s), \quad x \in \mathbb{R}, \operatorname{Re} s > 0.$$

The inverse Laplace transform, since  $\mathcal{L}^{-1} \left[ \frac{1+s^\alpha}{1+\tau s^\alpha} \right]$  is well-defined element in  $\mathcal{S}'_+$  (see [20]), gives

$$\sigma = \mathcal{L}^{-1} \left[ \frac{1 + s^\alpha}{1 + \tau s^\alpha} \right] *_t \varepsilon. \quad (13)$$

Setting  $L_t^\alpha = \mathcal{L}^{-1} \left[ \frac{1+s^\alpha}{1+\tau s^\alpha} \right] *_t$ , inserting  $\varepsilon$ , given by (8), into (13) and then inserting obtained  $\sigma$  into (6), we obtain (9). Note that  $L_t^\alpha = \mathcal{L}^{-1} \left[ \frac{1+s^\alpha}{1+\tau s^\alpha} \right] *_t$  can be explicitly expressed via the Mittag-Leffler function. Recall, for the Mittag-Leffler function  $e_\alpha$ , defined by

$$e_\alpha(t) = E_\alpha \left( -\frac{t^\alpha}{\tau} \right), \quad t > 0, \alpha \in (0, 1), \quad (14)$$

where  $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$ ,  $z \in \mathbb{C}$ , we have that  $e_\alpha \in C^\infty((0, \infty)) \cap C([0, \infty))$ ,  $e'_\alpha \in C^\infty((0, \infty)) \cap L^1_{loc}([0, \infty))$ , where  $e'_\alpha(t) = \frac{d}{dt} e_\alpha(t)$ ,  $t > 0$ , and

$$\mathcal{L}[e_\alpha(t)](s) = \frac{s^{\alpha-1}}{s^\alpha + \frac{1}{\tau}}, \quad \operatorname{Re} s > 0,$$

cf. [15]. Therefore,

$$\mathcal{L}^{-1} \left[ \frac{1 + s^\alpha}{1 + \tau s^\alpha} \right] (t) = \mathcal{L}^{-1} \left[ 1 + \frac{(1 - \tau)s^\alpha}{\tau(s^\alpha + \frac{1}{\tau})} \right] (t) = \frac{1}{\tau} \delta(t) + \left( \frac{1}{\tau} - 1 \right) e'_\alpha(t), \quad t > 0$$

and thus we obtain  $L_t^\alpha$  as given by (10).

For  $\alpha = 0$  and  $\beta = 1$ , i.e., when the Hooke law and the classical strain measure are used, equation (9) is the classical wave equation

$$\partial_t^2 u = c^2 \partial_x^2 u, \quad \text{with } c = \sqrt{\frac{2}{1 + \tau}}.$$

Therefore, system (6) - (8), or equivalently (9), generalize the classical wave equation. We collect other special cases of (9) in following remark.

**Remark 1** *Generalizations of the classical wave equation, given by system (6) - (8), or (9), are distinguished and classified according to parameter  $\beta$  as follows.*

- (i) *Case  $\beta = 0$ . We obtain the non-propagating disturbance if  $v_0 = 0$ . Namely, for  $\beta = 0$ , we obtain  $\varepsilon = 0$ , due to (8) and the property of the symmetrized fractional derivative that  $\mathcal{E}_x^0 u = 0$ , see Appendix A. This and (7), imply  $\sigma = 0$ , so that from (6), (11), and (12) one obtains*

$$u(x, t) = u_0(x) + v_0(x)t, \quad x \in \mathbb{R}, t \geq 0. \quad (15)$$

*Note, for  $v_0 = 0$ , we have  $u(x, t) = u_0(x)$ ,  $x \in \mathbb{R}$ ,  $t \geq 0$ .*

(ii) Case  $\beta \in (0, 1)$ . For  $\alpha = 0$  we obtain the space-fractional wave equation

$$\partial_t^2 u(x, t) = c^2 \partial_x \mathcal{E}_x^\beta u(x, t), \quad c = \sqrt{\frac{2}{1 + \tau}}, \quad x \in \mathbb{R}, \quad t > 0, \quad (16)$$

studied in [7]. Case  $\alpha \in (0, 1)$ , according to authors' knowledge, have not been studied in the literature, so it is the subject of analysis presented in this work. For  $\alpha = 1$ , (9) becomes the space-fractional Zener wave equation

$$\partial_t^2 u(x, t) = L_t^1 \partial_x \mathcal{E}_x^\beta u(x, t), \quad x \in \mathbb{R}, \quad t > 0. \quad (17)$$

For all  $\alpha \in [0, 1]$ , when  $\beta$  tends to zero, solution to system (6) - (8), (11), (12) tends to (15), see Section 3. This suggests that the parameter  $\beta$  measures the resistance of the material to the propagation of initial disturbance.

(iii) In the case when  $\beta = 1$ ,  $\alpha \in (0, 1)$ , equation (9) reduces to the time-fractional Zener wave equation

$$\partial_t^2 u(x, t) = L_t^\alpha \partial_x^2 u(x, t), \quad x \in \mathbb{R}, \quad t > 0, \quad (18)$$

studied in [14, 19]. For  $\alpha = 0$ , as already mentioned above, we obtain the classical wave equation and for  $\alpha = 1$  Zener wave equation

$$\partial_t^2 u(x, t) = L_t^1 \partial_x^2 u(x, t), \quad x \in \mathbb{R}, \quad t > 0.$$

## 2 Cauchy problem (9), (11)

### 2.1 Framework

The framework for our analysis are the spaces of distributions:  $\mathcal{S}'(\mathbb{R})$  (or shortly  $\mathcal{S}'$ ) and  $\mathcal{K}'(\mathbb{R})$  (or  $\mathcal{K}'$ ) the duals of the Schwartz space  $\mathcal{S}(\mathbb{R})$  (or  $\mathcal{S}$ ) and of the space  $\mathcal{K}(\mathbb{R})$  (or  $\mathcal{K}$ );  $\mathcal{K}$  is the space of smooth functions  $\varphi$  with the property  $\sup_{x \in \mathbb{R}, \alpha \leq m} |\varphi^{(\alpha)}(x)| e^{m|x|} < \infty$ ,  $m \in \mathbb{N}_0$ . The elements of  $\mathcal{S}'$ , respectively of  $\mathcal{K}'$ , are of the form  $f = \sum_{\alpha=0}^r \Phi_\alpha^{(\alpha)}$ , where  $\Phi_\alpha$  are continuous functions on  $\mathbb{R}$  and  $|\Phi_\alpha(t)| \leq C(1 + |t|)^{k_0}$ , respectively  $|\Phi_\alpha(t)| \leq C e^{k_0|t|}$ ,  $\alpha \leq r$ ,  $t \in \mathbb{R}$ , for some  $C > 0$ ,  $r \in \mathbb{N}_0$  and  $k_0 \in \mathbb{N}_0$ . The space  $\mathcal{S}'_+$  ( $\mathcal{K}'_+$ ) is a subspace of  $\mathcal{S}'$  ( $\mathcal{K}'$ ) consisting of elements supported by  $[0, \infty)$ . The elements of  $\mathcal{S}'_+$ , respectively of  $\mathcal{K}'_+$ , are of the form  $f(t) = (\Phi(t)(1 + |t|)^k)^{(p)}$ , respectively  $f(t) = (\Phi(t)e^{kt})^{(p)}$ ,  $t \in \mathbb{R}$ , where  $\Phi$  is a continuous bounded function such that  $\Phi(t) = 0$ ,  $t \leq 0$ . Note that  $\mathcal{S}'$  and  $\mathcal{S}'_+$  are subspaces of  $\mathcal{K}'$  and  $\mathcal{K}'_+$ , respectively. The elements of  $\mathcal{K}'_+$  have the Laplace transform, which are analytic functions in the domain  $\text{Re } s > s_0 > 0$ . We also recall that for the Lebesgue spaces of integrable and bounded functions  $L^1(\mathbb{R})$  and  $L^\infty(\mathbb{R})$ ,  $f * g \in L^\infty(\mathbb{R})$  if  $f \in L^1(\mathbb{R})$  and  $g \in L^\infty(\mathbb{R})$ .

We shall apply the Fourier transform with respect to  $x$  and the Laplace transform with respect to  $t$ . Actually, we shall consider the distributions within the space  $\mathcal{S}' \otimes \mathcal{K}'_+$ , which is the subspace of  $\mathcal{K}'(\mathbb{R}^2)$ , consisting of distributions having support in  $\mathbb{R} \times [0, \infty)$ . For the background of tensor product, we refer to [28]. We shall obtain the solution  $u$  as an element of  $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$  for fixed  $\varphi \in K$ , i.e.,  $\langle u(x, t), \varphi(t) \rangle \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$  and elements of  $K'_+$  for fixed  $\psi \in K$ , i.e.,  $\langle u(x, t), \psi(x) \rangle \in K'_+$ .

### 2.2 Existence and uniqueness of a generalized solution

We consider the existence and uniqueness of the solution to the Cauchy problem (9), (11), (12). If  $u_0 \in C^1(\mathbb{R})$  and  $v_0 \in C(\mathbb{R})$ , then the classical solution to the Cauchy problem (9), (11), (12) is

a function  $u(x, t)$  of class  $C^2$  for  $t > 0$ , of class  $C^1$  for  $t \geq 0$ , which satisfies equation (9) for  $t > 0$  and initial conditions (11) when  $t = 0$ , as well as the boundary conditions (12). If the function  $u$  is continued by zero for  $t < 0$ , then putting

$$u(x, t) = \mathcal{U}(x, t)H(t), \quad x, t \in \mathbb{R},$$

we obtain

$$\partial_t^2 \mathcal{U}(x, t) = L_t^\alpha \partial_x \mathcal{E}_x^\beta \mathcal{U}(x, t) + u_0(x) \delta'(t) + v_0(x) \delta(t), \quad \text{in } \mathcal{K}'(\mathbb{R}^2). \quad (19)$$

The main theorem is the following one.

**Theorem 2** *Let  $\alpha \in [0, 1)$ ,  $\beta \in [0, 1)$ ,  $\tau \in (0, 1)$  and let  $u_0, v_0 \in L^1(\mathbb{R})$ . Then there exists a unique generalized solution  $u \in \mathcal{K}'(\mathbb{R}^2)$ ,  $\text{supp } u \subset \mathbb{R} \times [0, \infty)$ , to the space-time fractional Zener wave equation (9), with initial (11) and boundary data (12).*

More precisely,  $u$  is of the form

$$u(x, t) = \frac{1}{2\pi^2} (\delta'(t) u_0(x) + \delta(t) v_0(x)) *_{x,t} P(x, t), \quad x \in \mathbb{R}, t > 0, \quad (20)$$

where

$$P(x, t) = I(x, t) - \left( \frac{\partial}{\partial t} J_1(x, t) + \frac{\partial^2}{\partial t^2} J_2(x, t) \right) e^{s_0 t}, \quad x \in \mathbb{R}, t > 0,$$

with

$$J_1 = i(J_1^+ - J_1^-), \quad J_2 = J_2^+ + J_2^-$$

and  $(x \in \mathbb{R}, t > 0)$

$$\begin{aligned} I(x, t) &= \int_{-p_0}^{p_0} \int_0^\infty \frac{\cos(\rho x) e^{s_0 t} e^{i p t}}{\left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta \pi}{2} \right]_{s=s_0+ip}} d\rho dp, \\ J_1^+(x, t) &= \int_{p_0}^\infty \int_0^1 \frac{\cos(\rho x) e^{i p t}}{p \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta \pi}{2} \right]_{s=s_0+ip}} d\rho dp, \\ J_1^-(x, t) &= \int_{p_0}^\infty \int_0^1 \frac{\cos(\rho x) e^{-i q t}}{q \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta \pi}{2} \right]_{s=s_0-iq}} d\rho dq, \\ J_2^+(x, t) &= \int_{p_0}^\infty \int_1^\infty \frac{\cos(\rho x) e^{i p t}}{p^2 \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta \pi}{2} \right]_{s=s_0+ip}} d\rho dp, \\ J_2^-(x, t) &= \int_{p_0}^\infty \int_1^\infty \frac{\cos(\rho x) e^{-i q t}}{q^2 \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta \pi}{2} \right]_{s=s_0-iq}} d\rho dq. \end{aligned}$$

Functions  $I, J_1^+, J_1^-, J_2^+$  and  $J_2^-$  are bounded and continuous functions with respect to  $x$  and continuous exponentially bounded functions with respect to  $t$ .

**Proof.** The plan of the proof is to solve (19) with the assumption that  $u_0$  and  $v_0$  are compactly supported smooth functions, i.e., elements of  $C_0^\infty(\mathbb{R})$ . Namely, if sequences  $\{u_{0n}\}_{n \in \mathbb{N}}, \{v_{0n}\}_{n \in \mathbb{N}} \in C_0^\infty(\mathbb{R})$  are such that  $u_{0n} \rightarrow u_0$  and  $v_{0n} \rightarrow v_0$  in  $L^1(\mathbb{R})$  as  $n \rightarrow \infty$ , then (20) is understood as

$$\begin{aligned} &(\delta'(t) u_0(x) + \delta(t) v_0(x)) *_{x,t} P(x, t) \\ &= \lim_{n \rightarrow \infty} ((\delta'(t) u_{0n}(x) + \delta(t) v_{0n}(x)) *_{x,t} P(x, t)) \quad \text{in } L^\infty(\mathbb{R}). \end{aligned}$$

Hence, (20), with  $u_0, v_0 \in L^1(\mathbb{R})$ , is a solution to (19). In the sequel, we assume that  $u_0, v_0 \in C_0^\infty(\mathbb{R})$ . This enables us to use the exchange formula.

Formally applying the Laplace transform to (9) with respect to  $t$ , with the initial conditions (11) taken into account, we obtain

$$\partial_x \mathcal{E}_x^\beta \tilde{u}(x, s) - s^2 \frac{1 + \tau s^\alpha}{1 + s^\alpha} \tilde{u}(x, s) = -\frac{1 + \tau s^\alpha}{1 + s^\alpha} (s u_0(x) + v_0(x)), \quad x \in \mathbb{R}, \operatorname{Re} s > s_0, \quad (21)$$

for suitably chosen  $s_0 > 0$ , where  $\tilde{u}$  is an analytic function with respect to  $s$ . Equation (21) is of the type

$$\partial_x \mathcal{E}_x^\beta u(x) - \omega u(x) = -\nu u_0(x) - \mu v_0(x), \quad x \in \mathbb{R}, \quad (22)$$

where

$$\omega = \omega(s) = s^2 \frac{1 + \tau s^\alpha}{1 + s^\alpha}, \quad \nu = \nu(s) = s \frac{1 + \tau s^\alpha}{1 + s^\alpha}, \quad \mu = \mu(s) = \frac{1 + \tau s^\alpha}{1 + s^\alpha}, \quad \operatorname{Re} s > s_0. \quad (23)$$

We have shown in [14, Theorem 4.2] that  $\omega(s) \in \mathbb{C} \setminus (-\infty, 0]$  for  $\operatorname{Re} s > 0$ . For fixed  $s$ ,  $\operatorname{Re} s > s_0$ , the unique solution  $u \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$  to (22), given by

$$u(x) = \frac{1}{\pi} (\nu u_0(x) + \mu v_0(x)) *_x \int_0^\infty \frac{1}{\rho^{1+\beta} \sin \frac{\beta\pi}{2} + \omega} \cos(\rho x) d\rho, \quad x \in \mathbb{R}, \quad (24)$$

is obtained as the inverse Fourier transform of

$$\hat{u}(\xi) = \frac{\nu \hat{u}_0(\xi) + \mu \hat{v}_0(\xi)}{|\xi|^{1+\beta} \sin \frac{\beta\pi}{2} + \omega}, \quad \xi \in \mathbb{R}.$$

The previous expression, with  $\omega$ ,  $\nu$  and  $\mu$  given by (23), takes the form

$$\hat{u}(\xi, s) = \frac{s \hat{u}_0(\xi) + \hat{v}_0(\xi)}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} |\xi|^{1+\beta} \sin \frac{\beta\pi}{2}}, \quad \xi \in \mathbb{R}, \operatorname{Re} s > s_0. \quad (25)$$

In fact, we have that  $x \mapsto \int_0^\infty \frac{1}{\rho^{1+\beta} \sin \frac{\beta\pi}{2} + \omega} \cos(\rho x) d\rho$  is a continuous bounded function ( $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ ) for fixed  $\omega = \omega(s)$ . After this function is convoluted with  $\nu u_0(x) + \mu v_0(x)$ , where  $u_0, v_0 \in L^1(\mathbb{R})$ , one obtains the function that belongs to  $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . Therefore, by (24), we have the solution to (21) in the form

$$\begin{aligned} \tilde{u}(x, s) &= \frac{1}{\pi} \frac{1 + \tau s^\alpha}{1 + s^\alpha} (s u_0(x) + v_0(x)) *_x \int_0^\infty \frac{1}{\rho^{1+\beta} \sin \frac{\beta\pi}{2} + s^2 \frac{1 + \tau s^\alpha}{1 + s^\alpha}} \cos(\rho x) d\rho \\ &= \frac{1}{\pi} (s u_0(x) + v_0(x)) *_x \int_0^\infty \frac{1}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \cos(\rho x) d\rho, \quad x \in \mathbb{R}, \operatorname{Re} s > s_0. \end{aligned} \quad (26)$$

Thus, the justification of the previously presented procedure is based on the analysis of the inverse Laplace transform. Formally, when applied to (26) the inverse Laplace transform gives

$$u(x, t) = \frac{1}{2\pi^2} (\delta'(t) u_0(x) + \delta(t) v_0(x)) *_x P(x, t), \quad x \in \mathbb{R}, t > 0, \quad (27)$$

where

$$P(x, t) = 2\pi \mathcal{L}^{-1} \left[ \int_0^\infty \frac{\cos(\rho x)}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} d\rho \right] (x, t) \quad (28)$$

$$= -i \int_{s_0 - i\infty}^{s_0 + i\infty} \int_0^\infty \frac{\cos(\rho x) e^{st}}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} d\rho ds, \quad x \in \mathbb{R}, t > 0. \quad (29)$$

Consider the divergent integral (29). We introduce the parametrization  $s = s_0 + ip$ ,  $p \in \mathbb{R}$ , in (29), so that

$$\begin{aligned} P(x, t) &= \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\cos(\rho x) e^{s_0 t} e^{ipt}}{\left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip}} d\rho dp \\ &= I(x, t) + I^+(x, t) + I^-(x, t), \quad x \in \mathbb{R}, t > 0, \end{aligned}$$

where  $I$ ,  $I^+$  and  $I^-$  are given below.

The integral

$$I(x, t) = \int_{-p_0}^{p_0} \int_0^{\infty} \frac{\cos(\rho x) e^{s_0 t} e^{ipt}}{\left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip}} d\rho dp, \quad x \in \mathbb{R}, t > 0,$$

is absolutely convergent, since

$$|I(x, t)| \leq e^{s_0 t} \int_{-p_0}^{p_0} \int_0^{\infty} \frac{1}{\operatorname{Re} \left( \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip} \right)} d\rho dp < \infty, \quad x \in \mathbb{R}, t > 0. \quad (30)$$

In (30),  $p_0$  is chosen so that

$$\begin{aligned} &\operatorname{Re} \left( \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip} \right) \\ &= r^2 \cos(2\varphi) + \frac{1 + (1+\tau)r^\alpha \cos(\alpha\varphi) + \tau r^{2\alpha}}{1 + 2\tau r^\alpha \cos(\alpha\varphi) + \tau^2 r^{2\alpha}} \rho^{1+\beta} \sin \frac{\beta\pi}{2} > 0, \end{aligned}$$

with  $r = \sqrt{s_0^2 + p^2}$  and  $\tan \varphi = \frac{p}{s_0}$ . Note that for  $p = 0$  and  $\rho = 0$  the integrand is well-defined, due to  $s_0 > 0$ . Thus, the integral  $I$  exists and belongs to  $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$  with respect to  $x$  and it is a continuous exponentially bounded function with respect to  $t$ . Next, we consider

$$\begin{aligned} I^+(x, t) &= e^{s_0 t} \int_{p_0}^{\infty} \int_0^{\infty} \frac{\cos(\rho x) e^{ipt}}{\left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip}} d\rho dp \\ &= -e^{s_0 t} \left( i \frac{\partial}{\partial t} J_1^+(x, t) + \frac{\partial^2}{\partial t^2} J_2^+(x, t) \right), \quad x \in \mathbb{R}, t > 0, \end{aligned} \quad (31)$$

where  $J_1^+$  and  $J_2^+$  are defined below. Setting

$$J_1^+(x, t) = \int_{p_0}^{\infty} \int_0^1 \frac{\cos(\rho x) e^{ipt}}{p \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip}} d\rho dp, \quad x \in \mathbb{R}, t > 0,$$

we have that the integral  $J_1^+$  exists and belongs to  $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$  with respect to both  $x$  and  $t$ , since

$$\begin{aligned} |J_1^+(x, t)| &\leq \int_{p_0}^{\infty} \int_0^1 \frac{1}{p \operatorname{Im} \left( \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip} \right)} d\rho dp \\ &\leq \frac{1}{2s_0} \int_{p_0}^{\infty} \int_0^1 \frac{1}{p^2} d\rho dp < \infty, \quad x \in \mathbb{R}, t > 0, \end{aligned}$$



where we used

$$\begin{aligned}
& \operatorname{Im} \left( \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip} \right) \\
&= r^2 \sin(2\varphi) + (1-\tau) \frac{r^\alpha \sin(\alpha\varphi)}{1+2\tau r^\alpha \cos(\alpha\varphi) + \tau^2 r^{2\alpha}} \rho^{1+\beta} \sin \frac{\beta\pi}{2} > 0 \\
&\sim r^2 \sin(2\varphi) + \frac{1-\tau}{\tau^2} \frac{1}{r^\alpha} \rho^{1+\beta} \sin(\alpha\varphi) \sin \frac{\beta\pi}{2}, \quad \text{as } r \rightarrow \infty \\
&\sim 2s_0 p + \frac{1-\tau}{\tau^2} \frac{1}{p^\alpha} \rho^{1+\beta} \sin \frac{\alpha\pi}{2} \sin \frac{\beta\pi}{2}, \quad \text{as } p \rightarrow \infty.
\end{aligned} \tag{32}$$

In obtaining (32) we used:  $r^2 \sin(2\varphi) = r^2 \frac{2 \tan \varphi}{1 + \tan^2 \varphi} = 2s_0 p$ , as well as  $\varphi \sim \frac{\pi}{2}$  and  $r \sim p$ , as  $p \rightarrow \infty$ . We put

$$J_2^+(x, t) = \int_{p_0}^{\infty} \int_1^{\infty} \frac{\cos(\rho x) e^{ipt}}{p^2 \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip}} d\rho dp, \quad x \in \mathbb{R}, t > 0,$$

which, by (32) and the Fubini theorem gives

$$\begin{aligned}
|J_2^+(x, t)| &\leq \int_{p_0}^{\infty} \int_1^{\infty} \frac{1}{p^2 \operatorname{Im} \left( \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip} \right)} d\rho dp \\
&\leq \int_{p_0}^{\infty} \left( \int_1^{\infty} \frac{1}{2s_0 p^3 + \frac{1-\tau}{\tau^2} p^{2-\alpha} \rho^{1+\beta} \sin \frac{\alpha\pi}{2} \sin \frac{\beta\pi}{2}} d\rho \right) dp \\
&\leq \frac{\tau^2}{1-\tau} \frac{1}{\sin \frac{\alpha\pi}{2} \sin \frac{\beta\pi}{2}} \int_{p_0}^{\infty} \frac{1}{p^{2-\alpha}} \left( \int_1^{\infty} \frac{1}{\rho^{1+\beta}} d\rho \right) dp < \infty, \quad x \in \mathbb{R}, t > 0.
\end{aligned}$$

By the same arguments as for  $J_1^+$ , we that the integral  $J_2^+$  exists and belongs to  $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$  with respect to both  $x$  and  $t$ . Thus, we have that  $I^+$ , given by (31), belongs to  $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$  with respect to  $x$ , and it is a derivative of a continuous exponentially bounded function with respect to  $t$ . Similarly as for  $I^+$ , we prove the existence for

$$\begin{aligned}
I^-(x, t) &= \int_{-\infty}^{-p_0} \int_0^{\infty} \frac{\cos(\rho x) e^{s_0 t} e^{ipt}}{\left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip}} d\rho dp \\
&= \int_{p_0}^{\infty} \int_0^{\infty} \frac{\cos(\rho x) e^{s_0 t} e^{-iqt}}{\left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0-iq}} d\rho dq \\
&= e^{s_0 t} \left( i \frac{\partial}{\partial t} J_1^-(x, t) - \frac{\partial^2}{\partial t^2} J_2^-(x, t) \right), \quad x \in \mathbb{R}, t > 0.
\end{aligned} \tag{33}$$

We have

$$J_1^-(x, t) = \int_{p_0}^{\infty} \int_0^1 \frac{\cos(\rho x) e^{-iqt}}{q \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0-iq}} d\rho dq, \quad x \in \mathbb{R}, t > 0,$$

so that

$$|J_1^-(x, t)| \leq \int_{p_0}^{\infty} \int_0^1 \frac{1}{q \left| \operatorname{Im} \left( \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0-iq} \right) \right|} d\rho dq, \quad x \in \mathbb{R}, t > 0.$$

The integral  $J_1^-$  exists and belongs to  $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$  with respect to both  $x$  and  $t$ , since

$$|J_1^-(x, t)| \leq \frac{1}{2s_0} \int_{p_0}^\infty \int_0^1 \frac{1}{q^2} d\rho dq < \infty, \quad x \in \mathbb{R}, \quad t > 0,$$

where  $r = \sqrt{s_0^2 + q^2}$ ,  $\tan \varphi = -\frac{q}{s_0}$  and where we used

$$\begin{aligned} & \operatorname{Im} \left( \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0-iq} \right) \\ &= r^2 \sin(2\varphi) + (1-\tau) \frac{r^\alpha \sin(\alpha\varphi)}{1+2\tau r^\alpha \cos(\alpha\varphi) + \tau^2 r^{2\alpha}} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \\ &\sim r^2 \sin(2\varphi) + \frac{1-\tau}{\tau^2} \frac{1}{r^\alpha} \rho^{1+\beta} \sin(\alpha\varphi) \sin \frac{\beta\pi}{2} < 0, \quad \text{as } r \rightarrow \infty \\ &\sim - \left( 2s_0 q + \frac{1-\tau}{\tau^2} \frac{1}{q^\alpha} \rho^{1+\beta} \sin \frac{\alpha\pi}{2} \sin \frac{\beta\pi}{2} \right), \quad \text{as } q \rightarrow \infty. \end{aligned}$$

Consider

$$J_2^-(x, t) = \int_{p_0}^\infty \int_1^\infty \frac{\cos(\rho x) e^{-iqt}}{q^2 \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0-iq}} d\rho dq, \quad x \in \mathbb{R}, \quad t > 0.$$

We have

$$\begin{aligned} |J_2^-(x, t)| &\leq \int_{p_0}^\infty \int_1^\infty \frac{1}{q^2 \left| \operatorname{Im} \left( \left[ s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0-iq} \right) \right|} d\rho dq \\ &\leq \int_{p_0}^\infty \left( \int_1^\infty \frac{1}{2s_0 q^3 + \frac{1-\tau}{\tau^2} q^{2-\alpha} \rho^{1+\beta} \sin \frac{\alpha\pi}{2} \sin \frac{\beta\pi}{2}} d\rho \right) dq \\ &\leq \frac{\tau^2}{1-\tau} \frac{1}{\sin \frac{\alpha\pi}{2} \sin \frac{\beta\pi}{2}} \int_{p_0}^\infty \frac{1}{q^{2-\alpha}} \left( \int_1^\infty \frac{1}{\rho^{1+\beta}} d\rho \right) dq, \quad x \in \mathbb{R}, \quad t > 0. \end{aligned}$$

The integral  $J_2^-$  exists and belongs to  $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$  with respect to both  $x$  and  $t$ . Thus, we have that  $I^-$ , given by (33), belongs to  $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$  with respect to  $x$ , and it is a derivative of a continuous exponentially bounded function with respect to  $t$ .

Thus,  $P$  has the form

$$P(x, t) = I(x, t) - \left( i \frac{\partial}{\partial t} (J_1^+(x, t) - J_1^-(x, t)) + \frac{\partial^2}{\partial t^2} (J_2^+(x, t) + J_2^-(x, t)) \right) e^{s_0 t}, \quad x \in \mathbb{R}, \quad t > 0,$$

where  $I$ ,  $J_1^+$ ,  $J_1^-$ ,  $J_2^+$  and  $J_2^-$  are bounded and continuous functions with respect to  $x$  and continuous exponentially bounded functions with respect to  $t$ . ■

### 2.3 Regularization of a generalized solution

We give a regularization of the generalized solution  $u$  to the space-time fractional Zener wave equation (19), which is of particular importance for the numerical analysis of the problem. We start from the Fourier and Laplace transform of the solution given by (25) and write it as

$$\widehat{u}(\xi, s) = \left( \widehat{u}_0(\xi) + \frac{1}{s} \widehat{v}_0(\xi) \right) \widehat{K}(\xi, s), \quad \xi \in \mathbb{R}, \quad \operatorname{Re} s > s_0, \quad (34)$$

where

$$\widehat{K}(\xi, s) = \frac{s}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} |\xi|^{1+\beta} \sin \frac{\beta\pi}{2}} = \frac{1}{s} - \widehat{Q}(\xi, s), \quad \text{with} \quad (35)$$

$$\widehat{Q}(\xi, s) = \frac{\frac{1+s^\alpha}{1+\tau s^\alpha} |\xi|^{1+\beta} \sin \frac{\beta\pi}{2}}{s^3 + s \frac{1+s^\alpha}{1+\tau s^\alpha} |\xi|^{1+\beta} \sin \frac{\beta\pi}{2}}, \quad \xi \in \mathbb{R}, \operatorname{Re} s > s_0. \quad (36)$$

Note that

$$\widehat{K}(\xi, s) = \frac{1}{2\pi} s \widehat{P}(\xi, s), \quad \text{i.e., } K(x, t) = \frac{1}{2\pi} \frac{\partial}{\partial t} P(x, t), \quad x, \xi \in \mathbb{R}, t > 0,$$

where  $P$  is given by (28). We already know from Theorem 2 that  $P$  (and therefore  $K$  as well) is a distribution. We regularize  $\widehat{K}$  by multiplying it with the Fourier transform of the Gaussian

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}} e^{-\frac{x^2}{\varepsilon^2}}, \quad x \in \mathbb{R}, \varepsilon \in (0, 1],$$

which is a  $\delta$ -net, i.e., Gaussian in a limiting process  $\varepsilon \rightarrow 0$  represents the Dirac delta distribution. Thus, we have that

$$\widehat{K}_\varepsilon(\xi, s) = \widehat{K}(\xi, s) e^{-\frac{(\varepsilon\xi)^2}{4}}, \quad \text{where } \mathcal{F}[\delta_\varepsilon(x)](\xi) = e^{-\frac{(\varepsilon\xi)^2}{4}}, \quad \xi \in \mathbb{R}, \operatorname{Re} s > s_0, \varepsilon \in (0, 1], \quad (37)$$

has the inverse Laplace and Fourier transforms which is a function and in a distributional limit gives the solution kernel  $K$  as a distribution.

We summarize these observations in the following theorem, given after we state the lemma.

**Lemma 3** *Let  $\alpha \in [0, 1)$ ,  $\tau \in (0, 1)$  and  $\theta > 0$ . Then*

$$\Psi_\alpha(s) = s^2 + \theta \frac{1+s^\alpha}{1+\tau s^\alpha}, \quad s \in \mathbb{C},$$

*admits exactly two zeros. They are complex-conjugate, located in the left complex half-plane and each of them is of the multiplicity one.*

**Theorem 4** *Let all conditions of Theorem 2 be satisfied. Let  $u \in \mathcal{K}'(\mathbb{R}^2)$ , with support in  $\mathbb{R} \times [0, \infty)$ , be generalized solution to the space-time fractional Zener wave equation (9), with initial (11) and boundary data (12). Then  $u$  is of the form:*

$$u(x, t) = (u_0(x)\delta(t) + v_0(x)H(t)) *_{x,t} K(x, t), \quad (38)$$

where  $K$  is a distributional limit in  $\mathcal{K}'(\mathbb{R}^2)$ :

$$K(x, t) = \lim_{\varepsilon \rightarrow 0} K_\varepsilon(x, t), \quad K_\varepsilon(x, t) = \frac{1}{\pi} \int_0^\infty S(\rho, t) \cos(\rho x) e^{-\frac{(\varepsilon\rho)^2}{4}} d\rho, \quad x \in \mathbb{R}, t > 0, \quad (39)$$

with

$$\begin{aligned} S(\rho, t) = & \frac{1}{2\pi i} \int_0^\infty \left( \frac{1}{q^2 + \frac{1+q^\alpha e^{i\alpha\pi}}{1+\tau q^\alpha e^{i\alpha\pi}} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} - \frac{1}{q^2 + \frac{1+q^\alpha e^{-i\alpha\pi}}{1+\tau q^\alpha e^{-i\alpha\pi}} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \right) q e^{-qt} dq \\ & + \frac{se^{st}}{2s + \frac{\alpha(1-\tau)s^{\alpha-1}}{(1+\tau s^\alpha)^2} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \Bigg|_{s=s_z(\rho)} + \frac{se^{st}}{2s + \frac{\alpha(1-\tau)s^{\alpha-1}}{(1+\tau s^\alpha)^2} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \Bigg|_{s=\bar{s}_z(\rho)}. \end{aligned} \quad (40)$$

and  $s_z$  are zeros of  $\Psi_\alpha$  from Lemma 3.

In particular, for suitable  $s_0 > 0$ ,  $K_\varepsilon(x, t) e^{-s_0 t}$  is bounded and continuous with respect  $x \in \mathbb{R}$ ,  $t > 0$ , for every  $\varepsilon \in (0, 1]$ .

**Proof of Lemma 3.** Let  $s = re^{i\varphi}$ ,  $r > 0$ ,  $\varphi \in (-\pi, \pi)$ . We have

$$\operatorname{Re} \Psi_\alpha(s) = r^2 \cos(2\varphi) + \theta \frac{1 + (1 + \tau)r^\alpha \cos(\alpha\varphi) + \tau r^{2\alpha}}{1 + 2\tau r^\alpha \cos(\alpha\varphi) + \tau^2 r^{2\alpha}}, \quad (41)$$

$$\operatorname{Im} \Psi_\alpha(s) = r^2 \sin(2\varphi) + \theta(1 - \tau) \frac{r^\alpha \sin(\alpha\varphi)}{1 + 2\tau r^\alpha \cos(\alpha\varphi) + \tau^2 r^{2\alpha}}. \quad (42)$$

From (41), (42) one can easily see that  $\Psi_\alpha(s_z) = 0$  implies  $\Psi_\alpha(\bar{s}_z) = 0$ .

Next we show that if  $\operatorname{Re} s_z > 0$ , then such  $s_z$  cannot be a zero of  $\Psi_\alpha$  and therefore zeros must lie in the left complex half-plane. Suppose  $\operatorname{Re} s > 0$ , i.e.,  $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . Since zeros appears in complex-conjugate pairs, we can suppose  $\varphi \in [0, \frac{\pi}{2})$ . For  $\varphi = 0$ , we have  $\Psi_\alpha(s) > 0$ . Since  $\alpha \in [0, 1)$ , we have that  $\alpha\varphi, 2\varphi \in (0, \pi)$  and therefore  $\sin(2\varphi) > 0$  and  $\sin(\alpha\varphi) > 0$ , which together with  $\theta > 0$  and  $\tau \in (0, 1)$  implies  $\operatorname{Im} \Psi_\alpha(s) > 0$ , so such  $s$  cannot be a zero.

It is left to show that there is only one pair of zeros of  $\Psi_\alpha$ . We use the argument principle. Recall, if  $f$  is an analytic function inside and on a regular closed curve  $C$ , and non-zero on  $C$ , then number of zeros of  $f$  (counted as many times as its multiplicity) inside the contour  $C$  is equal to the total change in the argument of  $f(s)$  as  $s$  travels around  $C$ . For our purpose we choose contour  $C = C_1 \cup C_2 \cup C_3 \cup C_4$ , parametrized as

$$\begin{aligned} C_1 : s &= xe^{i\frac{\pi}{2}}; \quad x \in [r, R], & C_2 : s &= Re^{i\varphi}; \quad \varphi \in \left[\frac{\pi}{2}, \pi\right], \\ C_3 : s &= xe^{i\pi}; \quad x \in [r, R], & C_4 : s &= re^{i\varphi}; \quad \varphi \in \left[\frac{\pi}{2}, \pi\right], \end{aligned}$$

where  $r < r_0$ ,  $R > R_0$  and  $r_0, R_0$  are chosen as follows:  $r_0$  is small enough such that for all  $r < r_0$  it holds that  $\operatorname{Re} \Psi(s) \sim \theta$  and  $\operatorname{Im} \Psi(s) \sim \theta(1 - \tau)r^\alpha \sin(\alpha\pi)$  and therefore there are no zeros for  $r < r_0$ , and  $R$  is large enough such that for all  $R > R_0$  it holds that  $\operatorname{Re} \Psi(s) \sim R^2 \cos(2\varphi)$  and  $\operatorname{Im} \Psi(s) \sim R^2 \sin(2\varphi)$ .

On the contour  $C_1$ , we have that  $\operatorname{Im} \Psi_\alpha(s) \geq 0$  (since  $\tau \in (0, 1)$  and  $\alpha \in [0, 1)$  implies  $\sin \frac{\alpha\pi}{2} \geq 0$ ), and  $\operatorname{Im} \Psi_\alpha(s) \rightarrow 0$  for  $r, x \rightarrow 0$ , as well as for  $R, x \rightarrow \infty$ . The real part of  $\Psi_\alpha$  varies from  $\theta$  (for  $r, x \rightarrow 0$ ) to  $-\infty$  (for  $R, x \rightarrow \infty$ ). Therefore, on  $C_1$  we have  $\Delta\Psi_\alpha(s) = -\pi$ .

On the contour  $C_2$ , for  $R > R_0$ ,  $R_0$  large enough, we have

$$\operatorname{Im} \Psi \sim R^2 \sin(2\varphi) + \frac{\theta(1 - \tau) \sin(\alpha\varphi)}{\tau^2} \frac{1}{R^\alpha} \sim R^2 \sin(2\varphi) \leq 0$$

and we have  $\operatorname{Im} \Psi_\alpha(s) \rightarrow 0$  for both  $\varphi = \frac{\pi}{2}$  and  $\varphi = \pi$ . The real part of  $\Psi_\alpha$  changes from  $-\infty$  (for  $\varphi = \frac{\pi}{2}$ ) to  $\infty$  (for  $\varphi = \pi$ ) since

$$\operatorname{Re} \Psi(s) \sim R^2 \cos(2\varphi) + \frac{\theta}{\tau} \sim R^2 \cos(2\varphi).$$

So, on  $C_2$  the change of the argument is  $\Delta\Psi_\alpha(s) = -\pi$ .

On the contours  $C_3$  and  $C_4$  argument does not change. On  $C_3$  imaginary part of  $\Psi_\alpha$  is always positive and it tends to zero for both  $R \rightarrow \infty$  and  $r \rightarrow 0$ , while real part changes from  $\infty$  (for  $R \rightarrow \infty$ ) to  $\theta$  (for  $r \rightarrow 0$ ), and even if it changes the sign it does not change the argument of  $\Psi_\alpha$ . On  $C_4$  it holds that  $\Psi_\alpha(s) \sim \theta$  and so there is no changes of argument.

Taking all together we have  $\Delta\Psi_\alpha(s) = -2\pi$  as  $s \in C$ , and by the argument principle there is one zero inside of the contour  $C$ . Therefore, there is a unique pair of complex-conjugate numbers in left complex plain which are zeros of  $\Psi_\alpha$ . ■

**Proof of Theorem 4.** Let

$$\widehat{K}_\varepsilon(\xi, s) = \frac{s}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} |\xi|^{1+\beta} \sin \frac{\beta\pi}{2}} e^{-\frac{(\varepsilon\xi)^2}{4}} = \frac{1}{s} e^{-\frac{(\varepsilon\xi)^2}{4}} - \widehat{Q}_\varepsilon(\xi, s), \quad \xi \in \mathbb{R}, \operatorname{Re} s > s_0, \quad (43)$$

by (35) and (37). We shall prove that

$$\widehat{\widehat{Q}}_\varepsilon(\xi, s) = \widehat{\widehat{Q}}(\xi, s) e^{-\frac{(\varepsilon\xi)^2}{4}} = \frac{\frac{1+s^\alpha}{1+\tau s^\alpha} |\xi|^{1+\beta} \sin \frac{\beta\pi}{2}}{s^3 + s \frac{1+s^\alpha}{1+\tau s^\alpha} |\xi|^{1+\beta} \sin \frac{\beta\pi}{2}} e^{-\frac{(\varepsilon\xi)^2}{4}}, \quad \xi \in \mathbb{R}, \operatorname{Re} s > s_0,$$

see (36), has the inverse Laplace and Fourier transforms by examining the convergence of the double integral ( $x \in \mathbb{R}$ ,  $t > 0$ ,  $\varepsilon \in (0, 1]$ )

$$\begin{aligned} Q_\varepsilon(x, t) &= \frac{1}{(2\pi)^2 i} \int_{s_0 - i\infty}^{s_0 + i\infty} \left( \int_{-\infty}^{\infty} \widehat{\widehat{Q}}_\varepsilon(\xi, s) e^{i\xi x} d\xi \right) e^{st} ds \\ &= \frac{1}{2\pi^2} (J_\varepsilon(x, t) + J_\varepsilon^+(x, t) + J_\varepsilon^-(x, t)) e^{s_0 t}, \end{aligned} \quad (44)$$

with ( $x \in \mathbb{R}$ ,  $t > 0$ ,  $\varepsilon \in (0, 1]$ )

$$J_\varepsilon(x, t) = \int_{-p_0}^{p_0} \int_0^\infty \frac{\frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}}{s^3 + s \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \Big|_{s=s_0+ip} \cos(\rho x) e^{-\frac{(\varepsilon\rho)^2}{4}} e^{ipt} d\rho dp, \quad (45)$$

$$J_\varepsilon^+(x, t) = \int_{p_0}^\infty \int_0^\infty \frac{\frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}}{s^3 + s \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \Big|_{s=s_0+ip} \cos(\rho x) e^{-\frac{(\varepsilon\rho)^2}{4}} e^{ipt} d\rho dp, \quad (46)$$

$$J_\varepsilon^-(x, t) = \int_{-\infty}^{-p_0} \int_0^\infty \frac{\frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}}{s^3 + s \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \Big|_{s=s_0+ip} \cos(\rho x) e^{-\frac{(\varepsilon\rho)^2}{4}} e^{ipt} d\rho dp, \quad (47)$$

where we introduced the parametrization  $s = s_0 + ip$ ,  $p \in (-\infty, \infty)$  in (44) and used the fact that  $\widehat{\widehat{Q}}_\varepsilon$  is an even function in  $\xi$ . From (45), we have ( $x \in \mathbb{R}$ ,  $t > 0$ ,  $\varepsilon \in (0, 1]$ )

$$|J_\varepsilon(x, t)| \leq \int_{-p_0}^{p_0} \int_0^\infty \frac{\left| \frac{1+s^\alpha}{1+\tau s^\alpha} \right|_{s=s_0+ip} \rho^{1+\beta}}{\left| \left[ s^3 + s \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip} \right|} e^{-\frac{(\varepsilon\rho)^2}{4}} d\rho dp < \infty.$$

Let us estimate the integral given by (46) as ( $x \in \mathbb{R}$ ,  $t > 0$ ,  $\varepsilon \in (0, 1]$ )

$$\begin{aligned} |J_\varepsilon^+(x, t)| &\leq \int_{p_0}^\infty \int_0^\infty \frac{\left| \frac{1+s^\alpha}{1+\tau s^\alpha} \right|_{s=s_0+ip} \rho^{1+\beta}}{\left| \left[ s^3 + s \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip} \right|} e^{-\frac{(\varepsilon\rho)^2}{4}} d\rho dp \\ &\leq \int_{p_0}^\infty \int_0^\infty \frac{\left| \frac{1+s^\alpha}{1+\tau s^\alpha} \right|_{s=s_0+ip} \rho^{1+\beta}}{\left| \operatorname{Im} \left( \left[ s^3 + s \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip} \right) \right|} e^{-\frac{(\varepsilon\rho)^2}{4}} d\rho dp. \end{aligned} \quad (48)$$

We have  $\operatorname{Im}(s_0 + ip)^3 = -p^3 + 3s_0^2 p$ ,  $\left| \frac{1+s^\alpha}{1+\tau s^\alpha} \right|_{s=s_0+ip} \sim \frac{1}{\tau}$  and

$$\operatorname{Im} \left( \left[ s \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2} \right]_{s=s_0+ip} \right) \sim \frac{1}{\tau} p + s_0 \frac{1-\tau}{\tau^2} \frac{1}{p^\alpha} \sin \frac{\alpha\pi}{2}, \quad \text{as } p \rightarrow \infty,$$

since

$$\begin{aligned}\operatorname{Re}\left(\frac{1+s^\alpha}{1+\tau s^\alpha}\right) &= \frac{1+(1+\tau)r^\alpha \cos(\alpha\varphi) + \tau r^{2\alpha}}{1+2\tau r^\alpha \cos(\alpha\varphi) + \tau^2 r^{2\alpha}} \sim \frac{1}{\tau}, \quad \text{as } r \rightarrow \infty, \\ \operatorname{Im}\left(\frac{1+s^\alpha}{1+\tau s^\alpha}\right) &= (1-\tau) \frac{r^\alpha \sin(\alpha\varphi)}{1+2\tau r^\alpha \cos(\alpha\varphi) + \tau^2 r^{2\alpha}} \sim \frac{1-\tau}{\tau^2} \frac{1}{r^\alpha} \sin(\alpha\varphi), \quad \text{as } r \rightarrow \infty,\end{aligned}$$

and thus for  $r = \sqrt{s_0^2 + p^2}$ ,  $\tan \varphi = \frac{s_0}{p}$

$$\begin{aligned}\operatorname{Im}\left(\left[s^3 + s \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}\right]_{s=s_0+ip}\right) \\ \sim -p^3 + 3s_0^2 p + \frac{1}{\tau} p + s_0 \frac{1-\tau}{\tau^2} \frac{1}{p^\alpha} \rho^{1+\beta} \sin \frac{\alpha\pi}{2} \sin \frac{\beta\pi}{2}, \quad \text{as } p \rightarrow \infty.\end{aligned}\quad (49)$$

We choose  $p_0$  so that (49) becomes

$$\operatorname{Im}\left(\left[s^3 + s \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}\right]_{s=s_0+ip}\right) \sim -p^3, \quad \text{as } p \rightarrow \infty,$$

Thus, for (48) we have

$$|J_\varepsilon^+(x, t)| \leq \int_{p_0}^\infty \int_0^\infty \frac{\rho^{1+\beta}}{p^3} e^{-\frac{(\varepsilon\rho)^2}{4}} d\rho dp < \infty.$$

Using the same arguments as for (46), we can prove that  $J_\varepsilon^-$ , given by (47), is also absolutely integrable.

We proved that  $Q_\varepsilon$ , given by (44), has the inverse Laplace and Fourier transforms and therefore, by (43), we have

$$\begin{aligned}\widehat{\hat{K}}_\varepsilon(\xi, s) &= \frac{1}{s} e^{-\frac{(\varepsilon\xi)^2}{4}} - \widehat{Q}_\varepsilon(\xi, s), \quad \xi \in \mathbb{R}, \operatorname{Re} s > s_0, \varepsilon \in (0, 1], \text{ i.e.,} \\ K_\varepsilon(x, t) &= H(t) \delta_\varepsilon(x) - Q_\varepsilon(x, t), \quad x \in \mathbb{R}, t > 0.\end{aligned}$$

Thus, in (43) we can first invert the Laplace transform and subsequently the Fourier transform. The Fourier transform of the solution kernel  $\hat{K}_\varepsilon$  is obtained by the use the inversion formula of the Laplace transform

$$\hat{K}_\varepsilon(\rho, t) = \frac{1}{2\pi i} \int_{s_0-i\infty}^{s_0+i\infty} \widehat{\hat{K}}_\varepsilon(\rho, s) e^{st} ds, \quad a \geq 0, \quad (50)$$

where  $\widehat{\hat{K}}_\varepsilon$  is given by (43) and the complex integration along the contour  $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_r \cup \Gamma_3 \cup \Gamma_4 \cup \gamma_0$ , presented in Figure 1. The contour  $\Gamma$  is parameterized by:

$$\begin{aligned}\Gamma_1 : s &= R e^{i\varphi}, \quad \varphi_0 < \varphi < \pi; \quad \Gamma_2 : s = q e^{i\pi}, \quad -R < -q < -r; \\ \Gamma_r : s &= r e^{i\varphi}, \quad -\pi < -\varphi < \pi; \quad \Gamma_3 : s = q e^{-i\pi}, \quad r < q < R; \\ \Gamma_4 : s &= R e^{i\varphi}, \quad -\pi < \varphi < \varphi_0; \quad \gamma_0 : s = s_0(1 + i \tan \varphi), \quad -\varphi_0 < \varphi < \varphi_0,\end{aligned}$$

for arbitrary chosen  $R > 0$  and  $0 < r < R$ , and  $\varphi_0 = \arccos \frac{s_0}{R}$ . By the Cauchy residues theorem and results of Lemma 3 we obtain:

$$\frac{1}{2\pi i} \oint_\Gamma \widehat{\hat{K}}_\varepsilon(\rho, s) e^{st} ds = \operatorname{Res}\left(\widehat{\hat{K}}_\varepsilon(\rho, s) e^{st}, s_z(\rho)\right) + \operatorname{Res}\left(\widehat{\hat{K}}_\varepsilon(\rho, s) e^{st}, \bar{s}_z(\rho)\right). \quad (51)$$

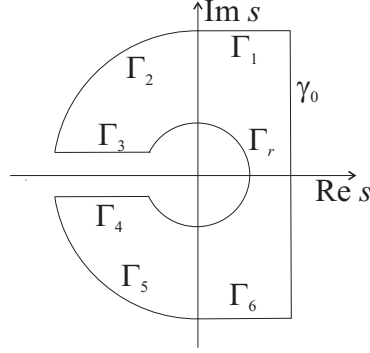


Figure 1: Integration contour  $\Gamma$ .

Now, one shows (see, for example, [14] for similar calculations) that in (51), when  $R$  tends to infinity and  $r$  tends to zero, integrals along contours  $\Gamma_1$ ,  $\Gamma_4$  and  $\Gamma_r$  tend to zero. The integrals along contours  $\Gamma_2$  and  $\Gamma_3$  in limiting process (when  $R$  tends to infinity and  $r$  tends to zero) read ( $\rho \geq 0$ ,  $t > 0$ )

$$\begin{aligned} \lim_{R \rightarrow \infty, r \rightarrow 0} \int_{\Gamma_2} \widehat{K}_\varepsilon(\rho, s) e^{st} ds &= -e^{-\frac{(\varepsilon\xi)^2}{4}} \int_0^\infty \frac{q}{q^2 + \frac{1+q^\alpha e^{i\alpha\pi}}{1+\tau q^\alpha e^{i\alpha\pi}} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} e^{-qt} dq, \\ \lim_{R \rightarrow \infty, r \rightarrow 0} \int_{\Gamma_3} \widehat{K}_\varepsilon(\rho, s) e^{st} ds &= e^{-\frac{(\varepsilon\xi)^2}{4}} \int_0^\infty \frac{q}{q^2 + \frac{1+q^\alpha e^{-i\alpha\pi}}{1+\tau q^\alpha e^{-i\alpha\pi}} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} e^{-qt} dq. \end{aligned}$$

By Lemma 3, we have that the residues in (51) read ( $\rho \geq 0$ ,  $t > 0$ )

$$\begin{aligned} \text{Res} \left( \widehat{K}_\varepsilon(\rho, s) e^{st}, s_z(\rho) \right) &= \frac{se^{st}}{2s + \frac{\alpha(1-\tau)s^{\alpha-1}}{(1+\tau s^\alpha)^2} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \Big|_{s=s_z(\rho)} e^{-\frac{(\varepsilon\xi)^2}{4}}, \\ \text{Res} \left( \widehat{K}_\varepsilon(\rho, s) e^{st}, \bar{s}_z(\rho) \right) &= \frac{se^{st}}{2s + \frac{\alpha(1-\tau)s^{\alpha-1}}{(1+\tau s^\alpha)^2} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \Big|_{s=\bar{s}_z(\rho)} e^{-\frac{(\varepsilon\xi)^2}{4}}. \end{aligned}$$

Integral along the contour  $\gamma_0$  in limiting process tends to the integral on the right-hand side of (50) and therefore, putting all together in (51) we obtain

$$\widehat{K}_\varepsilon(\rho, t) = S(\rho, t) e^{-\frac{(\varepsilon\rho)^2}{4}}, \quad x \in \mathbb{R}, \quad t > 0,$$

with  $S$  given by (40). The inverse Fourier transform of such obtained  $\widehat{K}_\varepsilon$  reads

$$K_\varepsilon(x, t) = \frac{1}{\pi} \int_0^\infty S(\rho, t) \cos(\rho x) e^{-\frac{(\varepsilon\rho)^2}{4}} d\rho, \quad x \in \mathbb{R}, \quad t > 0,$$

which in the distributional limit when  $\varepsilon \rightarrow 0$  gives the solution kernel  $K$  in the form (39). ■

### 3 Dependence of a solution on parameters $\alpha$ and $\beta$

We examine the solutions in the limiting cases of system (6) - (8), or equivalently (9), subject to (11), (12) in the view of Remark 1. In all cases we write the solution  $u$  of Theorem 4 as

$$u(x, t) = (u_0(x) \delta(t) + v_0(x) H(t)) *_{x,t} K_{\alpha,\beta}(x, t), \quad x \in \mathbb{R}, \quad t > 0,$$

where the inverse Fourier transform of  $\widehat{\tilde{K}}_{\alpha,\beta}$ , (35), is given by

$$\tilde{K}_{\alpha,\beta}(x, s) = \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \cos(\rho x) d\rho, \quad x \in \mathbb{R}, \operatorname{Re} s > 0. \quad (52)$$

Note that the integral in (52) is written formally and it denotes the inverse Fourier transform. As it will be seen, it may either converge, or diverge representing a distribution.

We are interested in the behavior of  $K_{\alpha,\beta}$  for  $\alpha$  and  $\beta$  tending to zero and one. We expect that the solution kernel  $K_{\alpha,\beta}$  tends to solution kernels in specific cases. From the form of  $K_{\alpha,\beta}$ , given by (39), this cannot be easily seen. However, numerical examples, see Section 4, suggest that this holds true. What can be seen analytically is that the Laplace transform of  $K_{\alpha,\beta}$  tends to the Laplace transforms of solution kernels in specific cases.

When  $\beta \rightarrow 0$ , then, in the sense of distributions,

$$\tilde{K}_{\alpha,0}(x, s) = \frac{1}{\pi} \int_0^\infty \frac{1}{s} \cos(\rho x) d\rho, \quad x \in \mathbb{R}, \operatorname{Re} s > 0,$$

and the solution kernel  $K_{\alpha,0}$  is of the form

$$K_{\alpha,0}(x, t) = \delta(x) H(t), \quad x \in \mathbb{R}, t > 0. \quad (53)$$

This is the case of the non-propagating disturbance, if the initial velocity is zero and the solution is given by (15) with  $v_0 = 0$ . Therefore, regardless of the parameter  $\alpha$ , when  $\beta$  tends to zero, solution kernel tends to (53). This supports the idea from Remark 1, (ii), that our system can be useful in modelling materials which resist the propagation of the initial disturbance.

When  $\beta \rightarrow 1$ , we obtain the case of the time-fractional Zener wave equation, studied in [14]. In this case

$$\tilde{K}_{\alpha,1}(x, s) = \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^2} \cos(\rho x) d\rho, \quad x \in \mathbb{R}, \operatorname{Re} s > 0,$$

and the calculation similar to one presented in [14], leads to

$$K_{\alpha,1}(x, t) = \frac{1}{4\pi i} \int_0^\infty \left( f_-(q) e^{|x|qf_-(q)} - f_+(q) e^{|x|qf_+(q)} \right) e^{-qt} dq, \quad x \in \mathbb{R}, t > 0, \quad (54)$$

with

$$f_+(q) = \sqrt{\frac{1 + \tau q^\alpha e^{i\alpha\pi}}{1 + q^\alpha e^{i\alpha\pi}}}, \quad \text{and} \quad f_-(q) = \sqrt{\frac{1 + \tau q^\alpha e^{-i\alpha\pi}}{1 + q^\alpha e^{-i\alpha\pi}}}, \quad q > 0.$$

We note that in [14] the solution is given in a slightly different form.

When  $\alpha \rightarrow 0$ , then

$$\tilde{K}_{0,\beta}(x, s) = \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + \frac{2}{1+\tau} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \cos(\rho x) d\rho, \quad x \in \mathbb{R}, \operatorname{Re} s > 0.$$

Using  $\mathcal{L}^{-1} \left[ \frac{s}{s^2 + \omega^2} \right] (t) = \cos(\omega t)$  one easily comes to

$$K_{0,\beta}(x, t) = \frac{1}{\pi} \int_0^\infty \cos \left( t \sqrt{\frac{2}{1+\tau} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \right) \cos(\rho x) d\rho, \quad x \in \mathbb{R}, t > 0,$$



in the sense of distributions, which can be transformed to

$$K_{0,\beta}(x, t) = \frac{1}{2\pi} \int_0^\infty \left( \cos \left( \left( x + ct \sqrt{\frac{1}{\rho^{1-\beta}} \sin \frac{\beta\pi}{2}} \right) \rho \right) + \cos \left( \left( x - ct \sqrt{\frac{1}{\rho^{1-\beta}} \sin \frac{\beta\pi}{2}} \right) \rho \right) \right) d\rho, \quad x \in \mathbb{R}, \quad t > 0, \quad (55)$$

where  $c = \sqrt{\frac{2}{1+\tau}}$ . This is the case of the space-fractional wave equation studied in [7]. Note that the solution in [7] is given in a different form. In this case, from (55), one can recover solution kernels for  $\beta = 0$  and  $\beta = 1$ .

If we put  $\beta = 0$  in (55), we obtain

$$K_{0,0}(x, t) = \frac{1}{\pi} \int_0^\infty \cos(x\rho) d\rho = \delta(x), \quad x \in \mathbb{R}, \quad t > 0,$$

i.e., (53).

For  $\beta = 1$  in (55), we obtain, in the sense of distributions,

$$\begin{aligned} K_{0,1}(x, t) &= \frac{1}{2\pi} \int_0^\infty (\cos((x+ct)\rho) + \cos((x-ct)\rho)) d\rho, \quad x \in \mathbb{R}, \quad t > 0, \\ &= \frac{1}{2} (\delta(x+ct) + \delta(x-ct)), \end{aligned}$$

with  $c = \sqrt{\frac{2}{1+\tau}}$ . This is the solution kernel for the classical wave equation.

**Remark 5 (Question of the wave speed.)** *Note that in all cases when  $\beta = 1$ , one obtains the finite and constant speed of wave propagation:  $c = \sqrt{\frac{2}{1+\tau}}$  in the case  $\alpha = 0$  and  $c = \frac{1}{\sqrt{\tau}}$  in the case  $\alpha \in (0, 1)$ . For  $\beta = 0$  the wave speed is zero. The case  $\beta \in (0, 1)$  is much more complicated for investigation. It seems that one would need to employ other technics, e.g. the theory of Fourier integral operators in order to reach some conclusions. However, we tend to believe that in such cases wave speed is not constant, depends on spatial variable  $x$  and parameter  $\beta$ , i.e.,  $c = c(x, \beta)$ .*

## 4 Numerical examples

We examine the qualitative properties of the solution to space-time fractional Zener wave equation (9). Further, we investigate the influence of the orders  $\alpha$  and  $\beta$  of the, respective, time and space fractionalization of the constitutive equation and strain measure. Also, we numerically compare solution to (9) with the solutions to time fractional Zener wave equation (18), that represents the limiting case  $\beta = 1$  in (9). Both equations are subject to initial conditions  $u_0 = \delta$ ,  $v_0 = 0$ . In this case, the solution to (9), given by (38), (39), becomes

$$u(x, t) = \delta(x) *_x K(x, t) = K(x, t), \quad x \in \mathbb{R}, \quad t > 0. \quad (56)$$

Since  $K$ , and therefore  $u$  is a distribution in  $x$ , it cannot be plotted. Thus, we use the regularization  $K_\varepsilon$  of the solution kernel so that (56) becomes

$$u_\varepsilon(x, t) = \frac{1}{\pi} \int_0^\infty \hat{K}(\rho, t) e^{-\frac{(\varepsilon\rho)^2}{4}} \cos(\rho x) d\rho, \quad x \in \mathbb{R}, \quad t > 0. \quad (57)$$

In all figures that are to follow, we present the displacement field only on the half-axis  $x \geq 0$ , since the field is symmetric with respect to displacement axis. Figure 2 presents the plot of the displacement versus coordinate, obtained according to (57) for several time instants, while the other parameters of the model are:  $\alpha = 0.25$ ,  $\beta = 0.45$ ,  $\tau = 0.1$ ,  $\varepsilon = 0.01$ . From Figure 2 we

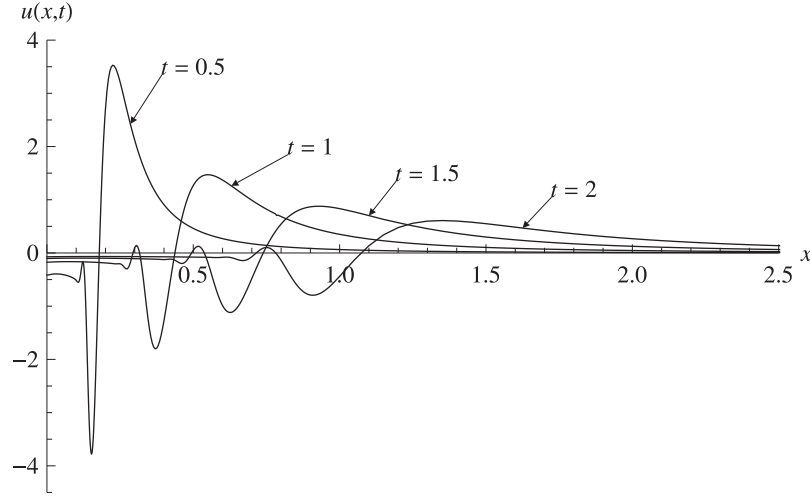


Figure 2: Displacement  $u(x, t)$  at  $t \in \{0.5, 1, 1.5, 2\}$  as a function of  $x$  as a solution of (9).

see that as time increases, the height of the peaks decreases, since the energy introduced by the initial disturbance field is being dissipated. This is the consequence of the viscoelastic properties of the material.

In Figure 3 we compared the displacements obtained as a solutions for non-local (9) and local (18) wave equations, given by (57) and (54), respectively. Apart from  $\beta = 0.45$  in (9) and  $\beta = 1$

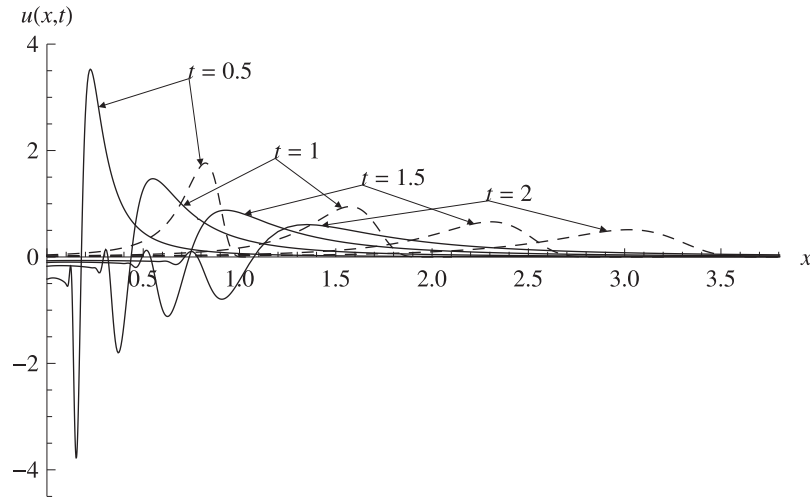


Figure 3: Displacement  $u(x, t)$  at  $t \in \{0.5, 1, 1.5, 2\}$  as a function of  $x$  as a solution of: (9) - solid line and (18) - dashed line.

in (18) other parameters in both models are as above. The effect of non-locality introduced in

the strain measure is observed, since at fixed time-instant, apart from the primary peak that exists in both models, in the non-local one there are secondary peaks in the displacement field. These secondary peaks reflect the influence of the oscillations of a certain material point to other material points in a medium. Thus, when the initial disturbance propagates (this is reflected by the existence of the primary peak), due to the non-locality, the secondary peaks reflect the residual influence of the disturbance transported by the primary peak. Not only that non-locality changes the number of peaks at a certain time-instant, but also the shape of the primary peak is changed. The primary peak in the non-local model (compared to the local one) is higher and placed closer to the origin - the point where the initial Dirac-type disturbance field is introduced.

The aim of the following figures is to show the influence of changing the non-locality parameter  $\beta$ . All other parameters are as above. Figure 4 presents plots of displacements for various values of  $\beta$ . From Figure 4 one sees that as the non-locality parameter  $\beta$  increases, the effects of non-

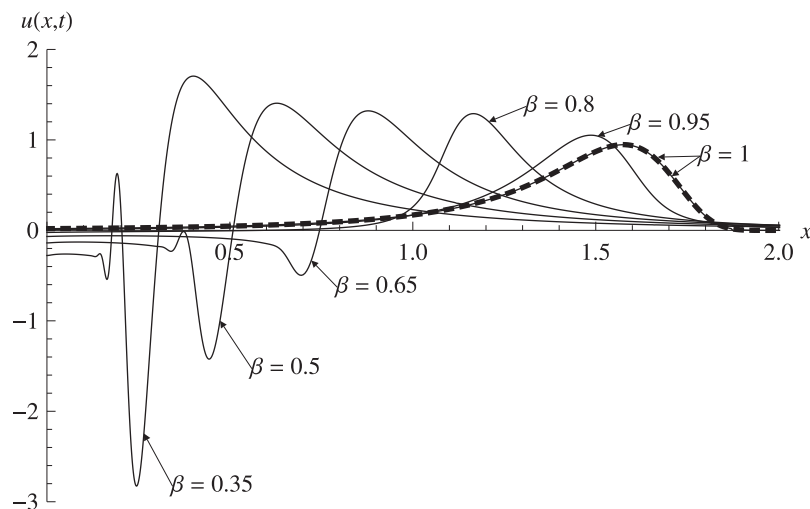


Figure 4: Displacement  $u(x, t)$  at  $t = 1$  as a function of  $x$  as a solution of: (9) - solid line and (18) - dashed line.

locality decrease, since the height of the secondary peaks decreases and eventually the secondary peaks cease to exist. Also, as  $\beta$  increases, the height of the primary peaks decreases and its position increases, being further from the point of the initial Dirac-type disturbance. In the limiting case  $\beta = 1$  the displacement curve of non-local model (9) overlaps with the displacement curve of the local model (18). Figures 5 and 6 present the displacement field for smaller values of  $\beta$ . In Figure 5 one notices the significant influence of non-local effects. Namely, the secondary peaks are more prominent in height than the primary peak. Finally, in the limiting case when  $\beta \rightarrow 0$  one expects to obtain the displacement field in the form (15), i.e.,  $u(x, t) = \delta(x)$ ,  $x \in \mathbb{R}$ ,  $t > 0$ . This can be seen from Figure 6. We might, thus, say that these numerical examples supports the claim that  $\beta$ , as the non-locality parameter, measures the resistance of material to the disturbance propagation. Namely, at the same time instant, as  $\beta$  decreases, the primary peak is placed closer to the point where the Dirac-type disturbance occurred and for  $\beta \rightarrow 0$  we obtain the non-propagating disturbance. Also, the shape of the primary peaks changes and, as  $\beta$  increases, they become more alike the peak of the local model and for  $\beta = 1$  we have the overlap of the curves.

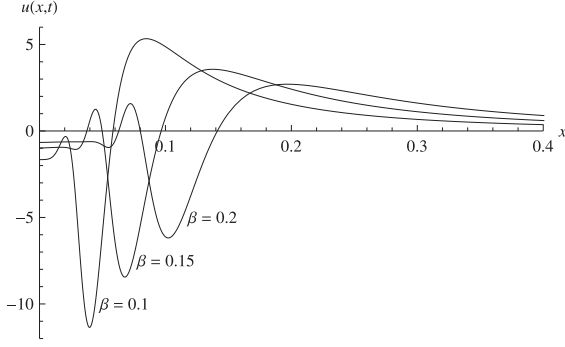


Figure 5: Displacement  $u(x, t)$  at  $t = 1$  as a function of  $x$  as a solution of (9).

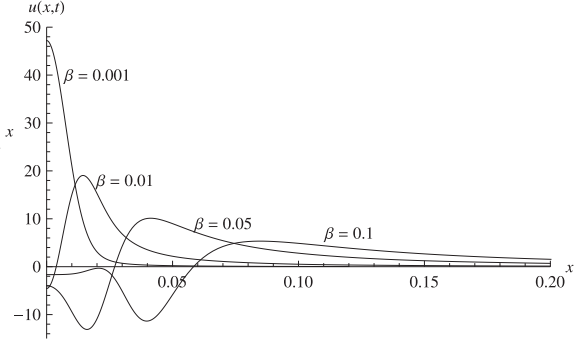


Figure 6: Displacement  $u(x, t)$  at  $t = 1$  as a function of  $x$  as a solution of (9).

## A Mathematical background

This section serves as a mathematical survey needed in analysis that we have presented. We single out definitions and properties of fractional derivatives and since our main tools are integral transforms, we recall, more or less well-known, main definitions and properties used. For a detailed exposition of the theory of fractional calculus see [24, 26], and for the spaces and integral transforms we refer to [26, 29].

Let  $0 \leq \alpha < 1$ ,  $-\infty \leq a < b \leq \infty$ . The left and right Caputo derivatives, of order  $\alpha$ , of an absolutely continuous function  $u$  are defined by

$${}_a^C D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{u'(\theta)}{(t-\theta)^\alpha} d\theta, \quad \text{and} \quad {}_t^C D_b^\alpha u(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b \frac{u'(\theta)}{(\theta-t)^\alpha} d\theta, \quad (58)$$

where  $\Gamma$  is the Euler gamma function and  $u' = \frac{d}{dt}u$ . Note that  ${}_a^C D_t^0 u(t) = {}_t^C D_b^0 u(t) = u(t)$ , and for continuously differentiable functions and distributions we have that as  $\alpha \rightarrow 1^-$ ,  ${}_a^C D_t^1 u(t) \rightarrow u'(t)$ ,  ${}_t^C D_b^1 u(t) \rightarrow -u'(t)$ . Therefore, the Caputo derivatives generalize integer order derivatives.

Let  $0 \leq \beta < 1$ ,  $-\infty \leq a < b \leq \infty$ . The symmetrized fractional derivative of an absolutely continuous function  $u$  is defined as

$$\begin{aligned} {}_a^C \mathcal{E}_b^\beta u(x) &= \frac{1}{2} \left( {}_a^C D_x^\beta - {}_x^C D_b^\beta \right) u(x) \\ &= \frac{1}{2} \frac{1}{\Gamma(1-\beta)} \int_a^b \frac{u'(\theta)}{|x-\theta|^\beta} d\theta. \end{aligned} \quad (59)$$

For  $a = -\infty$  and  $b = \infty$  we write  $\mathcal{E}_x^\beta$  instead of  ${}_a^C \mathcal{E}_b^\beta$  and then

$$\mathcal{E}_x^\beta u(x) = \frac{1}{2} \frac{1}{\Gamma(1-\beta)} |x|^{-\beta} * u'(x).$$

Note that  $\mathcal{E}_x^0 u(x) = 0$  and  $\mathcal{E}_x^\beta u(x) \rightarrow u'(x)$ , as  $\beta \rightarrow 1$ . So, the symmetrized fractional derivative generalizes the first derivative of a function. The zeroth order symmetrized fractional derivative of a function is zero (not a function itself).

For fractional operators in the distributional setting, one introduces a family  $\{f_\alpha\}_{\alpha \in \mathbb{R}} \in \mathcal{S}'_+$  as

$$f_\alpha(t) = \begin{cases} \frac{H(t) t^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0, \\ \frac{d^N}{dt^N} f_{\alpha+N}(t), & \alpha \leq 0, \alpha + N > 0, N \in \mathbb{N}, \end{cases}$$

and  $\{\check{f}_\alpha\}_{\alpha \in \mathbb{R}} \in \mathcal{S}'_-$  as

$$\check{f}_\alpha(t) = f_\alpha(-t),$$

where  $H$  is the Heaviside function. Then  $f_\alpha*$  and  $\check{f}_\alpha*$  are convolution operators and for  $\alpha < 0$  they are operators of left and right fractional differentiation, so that for  $u$  absolutely continuous we have

$${}_0^C D_t^\alpha u = f_{1-\alpha} * u' \quad \text{and} \quad {}_t^C D_a^\alpha u = -\check{f}_{1-\alpha} * u'.$$

For  $u \in \mathcal{S}'$  the Fourier transform is defined as

$$\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}),$$

where for  $\varphi \in \mathcal{S}$

$$\hat{\varphi}(\xi) = \mathcal{F}[\varphi(x)](\xi) = \int_{-\infty}^{\infty} \varphi(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{R}.$$

The Laplace transform of  $u \in \mathcal{S}'$  is defined by

$$\tilde{u}(s) = \mathcal{L}[u(t)](s) = \mathcal{F}[e^{-\xi t} u(t)](\eta), \quad s = \xi + i\eta.$$

It is well known that the function  $\tilde{u}$  is holomorphic in the half plane  $\operatorname{Re} s > 0$ , see e.g. [29]. In particular, for  $u \in L^1(\mathbb{R})$  such that  $u(t) = 0$ , for  $t < 0$ , and  $|u(t)| \leq Ae^{at}$  ( $a, A > 0$ ) the Laplace transform is

$$\tilde{u}(s) = \int_0^{\infty} u(t) e^{-st} dt, \quad \operatorname{Re} s > 0.$$

We recall main properties of the Fourier and Laplace transforms. Let  $u, u_1, u_2 \in \mathcal{S}'$

$$\begin{aligned} \mathcal{F}[u_1 * u_2](\xi) &= \mathcal{F}u_1(\xi) \cdot \mathcal{F}u_2(\xi), \quad \mathcal{F}[u^{(n)}](\xi) = (i\xi)^n \mathcal{F}u(\xi), \quad n \in \mathbb{N}, \quad \mathcal{F}\delta(s) = 1, \\ \mathcal{L}[u_1 * u_2](s) &= \mathcal{L}u_1(s) \cdot \mathcal{L}u_2(s), \quad \mathcal{L}[{}_0^C D_t^\alpha u](s) = s^\alpha \mathcal{L}u(s), \quad \alpha \geq 0, \quad \mathcal{L}\delta(s) = 1, \end{aligned}$$

where  $(\cdot)^{(n)}$  denotes  $n$ -th derivative. For  $\beta \in [0, 1)$  it holds

$$\begin{aligned} \mathcal{F}[|x|^{-\beta}](\xi) &= 2\Gamma(1-\beta) \sin \frac{\beta\pi}{2} \frac{1}{|\xi|^{1-\beta}}, \\ \mathcal{F}[\mathcal{E}_x^\beta u(x)](\xi) &= i \frac{\xi}{|\xi|^{1-\beta}} \sin \frac{\beta\pi}{2} \hat{u}(\xi). \end{aligned}$$

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